



# Stochastic nonlinear Schrödinger equations driven by a fractional noise well posedness, large deviations and support

Eric Gautier

## ► To cite this version:

Eric Gautier. Stochastic nonlinear Schrödinger equations driven by a fractional noise well posedness, large deviations and support. *Electronic Journal of Probability*, 2007, 12 (29), pp.848-861. hal-00102999

**HAL Id: hal-00102999**

**<https://hal.science/hal-00102999>**

Submitted on 3 Oct 2006

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

**STOCHASTIC NONLINEAR SCHRÖDINGER EQUATIONS DRIVEN  
BY A FRACTIONAL NOISE  
WELL POSEDNESS, LARGE DEVIATIONS AND SUPPORT**

ERIC GAUTIER

CREST - INSEE,  
3 avenue Pierre Larousse, 92240 Malakoff, France

IRMAR - Ecole Normale Supérieure de Cachan, Antenne de Bretagne,  
Campus de Ker Lann, avenue Robert Schuman, 35170 Bruz, France

ABSTRACT. In this article we consider stochastic nonlinear Schrödinger equations driven by an additive noise. The noise is fractional in time with Hurst parameter  $H$  in  $(0, 1)$ . It is also colored in space and the space correlation operator is assumed to be nuclear. We study the local well-posedness of the equations. Under adequate assumptions on the initial data, the space correlations of the noise and for some saturated nonlinearities, we prove a sample path large deviations principle and a support result. These results are stated in a space of exploding paths which are Hölder continuous in time until blow-up. We treat the case of Kerr nonlinearities when  $H > \frac{1}{2}$ .

*Key Words:* Stochastic partial differential equations, nonlinear Schrödinger equation, large deviations, fractional Brownian motion.

*AMS Subject Classification:* 60F10, 60H15, 35Q55.

## 1. INTRODUCTION

Nonlinear Schrödinger (NLS) equations are a generic model for the propagation of the envelope of a wave packet in weakly nonlinear and dispersive media, see [21]. They appear for example in nonlinear optics, hydrodynamics, biology, field theory, crystals, Bose-Einstein condensates, Fermi-Pasta-Ulam chains of atoms. Sometimes random perturbations have to be considered. In optics, noise accounts for the spontaneous emission noise due to amplifiers placed along the fiber line in order to compensate for loss in the fiber. In the context of crystals or of Fermi-Pasta-Ulam chains of atoms, noise sometimes accounts for thermal effects. Noises considered are often either complex additive noises or real multiplicative noises. In physics, the Gaussian space-time white noise is broadly considered. It has not been possible so far to give a mathematical meaning to the solutions of such equations. Noises considered in mathematics are colored in space. Note that in optics, because the time variable corresponds to space and the space variable to some retarded time, noises considered for well-posedness are indeed colored in time.

We consider here the case of a fractional additive noise. Fractional noises, introduced by Mandelbrodt, have several applications in hydrology, finance and telecommunications. They are extensions of the Gaussian white noise ( $H = \frac{1}{2}$ ) and for  $H \neq \frac{1}{2}$  the noises are colored in time. Up to our knowledge, these noises have not been considered in Physics for such models. Again, in optics the new correlations could account for correlations in space. However, we consider such noises to show that the results of [13] can be extended to more general Gaussian noises. We specify these particular fractional noises for computational issues since we then know precisely the kernel.

The stochastic NLS equations could be written with the Itô notations

$$(1.1) \quad i du - (\Delta u + f(u))dt = dW^H,$$

where  $u$  is a complex valued function of time and space and  $W^H$  is a fractional Wiener process. The fractional noise is formally its time derivative. The parameter  $H$  is called the Hurst parameter. It belongs to  $(0, 1)$ . The space variables belong to the whole space  $\mathbb{R}^d$ . The initial datum  $u_0$  is a function of a particular Sobolev space based on  $L^2$ .

We consider pathwise weak solutions in the sense used in the analysis of PDEs. More precisely, we are interested in mild solutions which are such that

$$(1.2) \quad u(t) = U(t)u_0 - i \int_0^t U(t-s)f(u(s))ds - i \int_0^t U(t-s)dW^H(s),$$

where  $(U(t))_{t \in \mathbb{R}}$  is the Schrödinger linear group on some Sobolev space  $H^s$  generated by the skew-adjoint unbounded operator  $(-i\Delta, H^{s+2})$ .

Nonlinearities of the form  $f(u) = \lambda|u|^{2\sigma}u$  where  $\lambda = \pm 1$  are often considered for NLS equations, they are called Kerr nonlinearities. In that case the space of energy  $H^1$  is of particular interest. It is such that the Hamiltonian is well defined. It is also a space where the blow-up phenomenon is usually studied, indeed localized functions and thus the variance may be defined. These nonlinearities are Lipschitz on the bounded sets of  $H^1$  iff  $d = 1$ . In higher dimensions, the Strichartz inequalities, see [21] allow to treat these nonlinearities. For the stochastic equations driven by a Gaussian noise which is white in time, it is proved in [6] that the Cauchy problem is locally well-posed in  $H^1$  for every  $\sigma$  when  $d = 2$  and only when  $\sigma < \frac{2}{d-2}$  for  $d \geq 3$ . Also, for such values of  $\sigma$  and  $\lambda = -1$ , defocusing case, the Cauchy problem is globally well-posed. It is proved considering the mass and Hamiltonian which are invariant quantities of the deterministic equation. In the focusing case when  $\lambda = 1$ , solutions may blow-up in finite time when

$\sigma \geq \frac{2}{d}$ , critical and supercritical nonlinearities. Note that in [7], theoretical results on the influence of a noise on the blow-up phenomenon have been obtained. Large deviations and a support theorem for such equations is given in [13]. In [14], we prove LDP for a noise of multiplicative type. In [10, 13], we apply our results to the problem of error in soliton transmission by analyzing the optimal control problem that governs the rate of the exponential decay to zero with the noise intensity of the probability of large deviation events. Note that, for the fractional noise of this article, the computations would then be almost untractable because of the extra correlations in time. Also, in [15], we apply the uniform LDPs to the study the problem of the exit from a domain of attraction for weakly damped equations. We use the strong Markov property which does not hold for fractional noises. We also use LDPs to obtain estimates on the small noise asymptotic of the exit times in [13, 14].

A fractional Brownian motion (fBm) is a centered Gaussian processes with stationary increments

$$\mathbb{E} \left( |\beta^H(t) - \beta^H(s)|^2 \right) = |t - s|^{2H}, \quad t, s > 0.$$

A cylindrical fractional Wiener process on a Hilbert space consists formally of independent fractional Brownian motions (fBm) (with possibly different Hurst parameters) on each coordinate of a complete orthonormal system. It does not have trajectories in the Hilbert space. Also, only images by Hilbert-Schmidt mappings are such that the laws of the marginals are *bona fide* Radon measures. We cannot expect the stochastic convolution to make sense removing the Hilbert-Schmidt assumption since the group has no global smoothing properties in the Sobolev spaces based on  $L^2$ . It is an isometry on such spaces. Also, since we work in  $\mathbb{R}^d$ , the stochastic convolution would have to be space wise translation invariant which is not compatible with the fact that it should be a process with paths in a Sobolev space based on  $L^2$ . Thus we assume that the fractional Wiener processes in our equation is a direct image via a Hilbert-Schmidt operator of a cylindrical fractional Wiener process on  $L^2$ .

In this article, we consider the semi-group approach developped in [5]. It is well suited for stochastic NLS equations where we use properties of this group. It also allows to define the stochastic integration in infinite dimensions with the well studied integration with respect to the one parameter and one dimensional fBm. More precisely we use the approach to the stochastic calculus with respect to the fractional Brownian motion developed in [1] for general Voltera processes which is based on the Malliavin calculus. In that case, the stochastic integral is a Skohorod integral.

We prove that the Cauchy problem is locally well-posed. We then prove, for particular saturated nonlinearities *i.e.* nonlinearities which are locally Lipschitz, a sample path large deviation principle (LDP) for the small noise asymptotic and a support theorem in a space of exploding paths which are  $H'$ -Hölder continuous on time intervals before blow-up with  $0 < H' < H$ . Though the Hölder regularity holds for the stochastic convolution it cannot be transfered easily since the group is an isometry. We impose additional regularity of the initial datum and suitable assumptions on the correlations in space of the noise. In the last section we treat the case of the Kerr nonlinearities for  $H > \frac{1}{2}$  but do not impose conditions to obtain Hölder continuous paths.

It is certainly much more involved to treat multiplicative noises. For example, the stochastic convolution is now anticipating. It is for the same reason that we do not investigate the global existence. Indeed, the Itô formula applied to the Hamiltonian and mass to a certain power as in [6] gives rise to anticipating stochastic integrals. These questions will be studied in future works.

## 2. PRELIMINARIES

The space of complex Lebesgue square integrable functions  $L^2$  with the inner product defined by  $(u, v)_{L^2} = \Re \int_{\mathbb{R}} u(x)\bar{v}(x)dx$  is a Hilbert space. For  $r$  positive, the Sobolev spaces  $H^r$  are the Hilbert spaces of functions  $f$  of  $L^2$  such that their Fourier transform  $\hat{f}$  satisfy  $\int_{\mathbb{R}^d} (1 + |\xi|^2)^r |\hat{f}(\xi)|^2 d\xi < \infty$ . If  $I$  is an interval of  $\mathbb{R}$ ,  $(E, \|\cdot\|_E)$  a Banach space and  $r$  belongs to  $[1, \infty]$ , then  $L^r(I; E)$  is the space of strongly Lebesgue measurable functions  $f$  from  $I$  into  $E$  such that  $t \rightarrow \|f(t)\|_E$  is in  $L^r(I)$ . The integral is the Bochner integral. The space of bounded operators from  $B$  to  $C$ , two Banach spaces, is denoted by  $\mathcal{L}_c(B, C)$ . The space of Hilbert-Schmidt operators  $\Phi$  from  $E$  to  $F$ , two Hilbert spaces, is denoted by  $\mathcal{L}_2(E, F)$ . It is a Hilbert space when endowed with the norm  $\|\Phi\|_{\mathcal{L}_2(E, F)}^2 = \text{tr} \Phi \Phi^* = \sum_{j \in \mathbb{N}} \|\Phi e_j\|_F^2$  where  $(e_j)_{j \in \mathbb{N}}$  is a complete orthonormal system of  $E$ . We denote by  $\mathcal{L}_2^{0, r}$  the above space when  $E = L^2$  and  $F = H^r$ .

When  $A$  and  $B$  are two Banach spaces,  $A \cap B$  with the norm defined as the maximum of the norms in  $A$  and in  $B$ , is a Banach space. A pair  $(r, p)$  of positive numbers is called an admissible pair if  $p$  satisfies  $2 \leq p < \frac{2d}{d-2}$  when  $d > 2$  ( $2 \leq p < +\infty$  when  $d = 2$  and  $2 \leq p \leq +\infty$  when  $d = 1$ ) and  $r$  is such that  $\frac{2}{r} = d \left( \frac{1}{2} - \frac{1}{p} \right)$ . Given an admissible pair  $(r(p), p)$  and  $T$  positive, the space

$$X^{(T, p)} = C([0, T]; H^1) \cap L^{r(p)}(0, T; W^{1, p}),$$

is the space considered to prove the local existence of solutions to the NLS equation with a Kerr nonlinearity.

Also, we denote by  $C^{H'}([0, T]; E)$  the space of  $H'$ -Hölder  $E$ -valued continuous functions on  $[0, T]$  embedded with the norm

$$\|f\|_{H', T} = \sup_{t \in [0, T]} \|f(t)\|_E + \sup_{t, s \in [0, T], t \neq s} \frac{\|f(t) - f(s)\|_E}{|t - s|^{H'}}$$

where  $E$  is a Banach space. The space  $C^{H', 0}([0, T]; E)$  is the separable subset of the above such that

$$\lim_{|t-s| \rightarrow 0} \frac{\|f(t) - f(s)\|_E}{|t - s|^{H'}} = 0.$$

We denote by  $x \wedge y$  the minimum of  $x$  and  $y$ . A rate function  $I$  is a lower semicontinuous function. It is good if for every  $c$  positive,  $\{x : I(x) \leq c\}$  is compact.

Volterra processes, see for example [8], are defined for  $T$  positive as

$$X(t) = \int_0^t K(t, s) d\beta(s), \quad K \in L^2([0, T] \times [0, T]), \quad T > 0, \quad K(t, s) = 0 \text{ if } s > t.$$

The covariance of such a process is

$$R(t, s) = \int_0^{t \wedge s} K(t, r) K(s, r) dr.$$

The covariance operator, when we consider the  $L^2(0, T)$ -random variables, has finite trace. It could be defined through the kernel  $R(t, s)$ , *i.e.* for  $h$  in  $L^2(0, T)$ ,  $Rh(t) = \int_0^T R(t, s) h(s) ds$ . Also  $R$  is such that  $R = KK^*$  where  $K$  is the Hilbert-Schmidt operator defined for  $h \in L^2(0, T)$  by  $Kh(t) = \int_0^T K(t, s) h(s) ds = \int_0^t K(t, s) h(s) ds$  and  $K^*$  is its adjoint. These processes admit modifications with continuous sample paths; they are Gaussian processes.  $\text{im } R^{\frac{1}{2}}$ , the range of  $R^{\frac{1}{2}}$  with the norm of the image structure, is the

reproducing kernel Hilbert space (RKHS) of the Gaussian measure which is the law of the process on  $L^2(0, T)$ . It is classical that is equal to  $\text{im } K$ . It is also the RKHS of the measure on  $C([0, T])$  since the restriction of the measure is again a Gaussian probability measure and  $C([0, T])$  is a Banach space continuously embedded in  $L^2(0, T)$ . Also, it is known that the RKHS of the measure on  $C([0, T])$  is isometric to the closure in  $L^2(\mu)$ , where  $\mu$  is the Gaussian measure, of the dual of the Banach space defined by means of the evaluation at points  $t$  in  $[0, T]$  and thus to the first Wiener chaos.

It is now standard fact, see for example [3, 12], that the rate function of a LDP for the family of Gaussian measures defined as direct images of  $\mu$  via the mapping  $x \mapsto \sqrt{\epsilon}x$  is given by  $\frac{1}{2} \|\cdot\|_{\text{im } R^{\frac{1}{2}}}^2$  and that the support of the law of the Gaussian measure is the closure of the RKHS for the norm of the Banach space. We aim to transport such results to the law of the mild solution of the stochastic NLS equation driven by such noises. We know that it is more convenient to primarily prove large deviations for a modification of the infinite dimensional stochastic convolution with smooth sample paths.

Defining the stochastic integration in Hilbert spaces only requires to define the stochastic integration in dimension one. Such Volterra processes are seldom martingales. However, since they are Gaussian processes, the Skorohod integral can be defined. Let us recall, for the sake of completeness, some aspects of integration with respect to Volterra processes with the Malliavin calculus, see [1] for more details. Let us first consider another space well suited for the Malliavin calculus that is also a RKHS. It may be seen as generated by step functions on  $[0, T]$ ; the stochastic integral of a step function  $\mathbb{1}_{[0, t]}$  should coincide with the evaluation at point  $t$ . The set of step functions is denoted by  $\mathcal{E}$ . We consider the inner product defined by

$$R(t, s) = \langle \mathbb{1}_{[0, t]}, \mathbb{1}_{[0, s]} \rangle_{\mathcal{H}} = (K(t, \cdot) \mathbb{1}_{[0, t]}, K(s, \cdot) \mathbb{1}_{[0, s]})_{L^2(0, T)}.$$

The linear operator  $K_T^*$  from  $\mathcal{E}$  into  $L^2(0, T)$  is defined for  $\varphi$  in  $\mathcal{E}$  by

$$(2.1) \quad (K_T^* \varphi)(s) = \varphi(s)K(T, s) + \int_s^T (\varphi(t) - \varphi(s))K(dt, s).$$

It is such that, for any  $\varphi$  in  $\mathcal{E}$  and  $h$  in  $L^2(0, T)$ , we have

$$\int_0^T (K_T^* \varphi)(t)h(t)dt = \int_0^T \varphi(t)(Kh)(dt).$$

The RKHS  $\mathcal{H}$  is now obtained as the closure of  $\mathcal{E}$  with respect to the norm  $\|\varphi\|_{\mathcal{H}} = \|K_T^* \varphi\|_{L^2(0, T)}$ . The operator  $K_T^*$  is then an isometry between  $\mathcal{H}$  and a closed subspace of  $L^2(0, T)$ ; we represent  $\mathcal{H}$  as  $\mathcal{H} = (K_T^*)^{-1}(L^2(0, T))$ . The above duality relation allows to extend integration with respect to  $Kh(dt)$  to integrands in  $\mathcal{H}$ . It also allows to define a stochastic integration with respect to these Volterra processes for integrands  $\varphi$  in  $\mathcal{H}$  as the Skorohod integral

$$\delta^X(\varphi) = \int_0^T (K_T^* \varphi)(t)\delta\beta(t).$$

For deterministic integrands, it is the case for the stochastic convolution with an additive noise, the integral could be written as a Itô integral

$$\delta^X(\varphi) = \int_0^T (K_T^* \varphi)(t)d\beta(t).$$

From now on we restrict our attention to the particular case of the fBm. Enlarging if necessary the probability space the fBm may be defined in terms of a standard Brownian

motion  $(\beta(t))_{t \geq 0}$  via the square integrable triangular kernel  $K^H$ , i.e.  $K^H(t, s) = 0$  if  $s > t$ ,

$$\beta^H(t) = \int_0^t K^H(t, s) d\beta(s),$$

where

$$(2.2) \quad K^H(t, s) = c_H(t-s)^{H-\frac{1}{2}} + c_H \left( \frac{1}{2} - H \right) \int_s^t (u-s)^{H-\frac{3}{2}} \left( 1 - \left( \frac{s}{u} \right)^{\frac{1}{2}-H} \right) du,$$

and

$$c_H = \left( \frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)} \right)^{\frac{1}{2}}.$$

(2.2) implies that

$$(2.3) \quad \frac{\partial K^H}{\partial t}(t, s) = c_H \left( \frac{1}{2} - H \right) (t-s)^{H-\frac{3}{2}} \left( \frac{s}{t} \right)^{\frac{1}{2}-H}.$$

We now denote the kernel and the operator by  $K$  instead of  $K^H$  for the fBm.

Also, we recall the following properties. The fBm has a modification with  $H'$ -Hölder continuous sample paths where  $0 < H' < H$ ; see for example [9]. Its covariance is given by

$$\mathbb{E}(\beta^H(t)\beta^H(s)) = \frac{1}{2} (s^{2H} + t^{2H} - |s-t|^{2H}).$$

The increments are independent if and only if  $H = \frac{1}{2}$ . The covariance of future and past increments is negative if  $H < \frac{1}{2}$  and positive if  $H > \frac{1}{2}$ . Thus we say that the fBm presents long range dependence for  $H > \frac{1}{2}$  as the covariance between increments at a distance  $u$  decays as  $u^{2H-2}$ . Finally note that these processes are also self-similar, i.e. the law of the paths  $t \mapsto \beta^H(at)$  where  $a$  is positive are that of  $t \mapsto a^H \beta^H(t)$ . The solution of the NLS equation also display a self similar behavior, but in the space variable, near blow-up for supercritical nonlinearities, see [21].

In the particular case of the fBm the stochastic integration may also be defined by means of the fractional calculus. A rough paths approach may also be considered, see for example [4]. We expect that this latter approach could allow to treat noises of multiplicative type.

**Remark 2.1.** *Suppose that for multiplicative noises we were able to prove the continuity of the solution with respect to the driving process at the level of the rough paths, see [19] for certain SDEs, then LDP and support theorems follow from a contraction principle stating large deviations for the rough paths of the driving process. It is done in [17] for a SDE driven by the Brownian motion. LDP for the rough paths of the fBm and for a Banach space valued Wiener process are proved in [20] and [16]. Also a rough paths approach to a linear SPDE with analytical semigroup and for "smooth" rough paths is given in [18].*

We use several times the following property, that we may check using (2.1) and (2.2), that for  $0 < t < T$ ,

$$(2.4) \quad (K_T^* \mathbb{1}_{[0,t]} \varphi)(s) = (K_t^* \varphi)(s) \mathbb{1}_{[0,t]}(s).$$

For smooth kernels such that  $H > \frac{1}{2}$ , relation (2.1) has the simpler form

$$(2.5) \quad (K_T^* \varphi)(s) = \int_s^T \varphi(r) K(dr, s).$$

The formulation in (2.1) however allows to extend this definition to singular kernels, *i.e.* when  $H < \frac{1}{2}$ . For  $H$  such that  $H > \frac{1}{2}$ , the inner product in  $\mathcal{H}$  of  $\varphi$  and  $\psi$  is given by

$$\begin{aligned} \langle \varphi, \psi \rangle_{\mathcal{H}} &= \int_0^T \int_0^T \varphi(u) \psi(v) \int_0^{u \wedge v} \frac{\partial K}{\partial u}(u, s) \frac{\partial K}{\partial v}(v, s) ds du dv \\ &= c_H^2 \left(H - \frac{1}{2}\right)^2 B\left(2 - 2H, H - \frac{1}{2}\right) \int_0^T \int_0^T \varphi(u) \psi(v) |u - v|^{2H-2} du dv, \end{aligned}$$

from a computation given in [2];  $B$  denotes the Beta function. It corresponds to the covariance of the stochastic integrals with respect to the fBm

$$\mathbb{E} \left[ \int_0^T \varphi(u) d\beta^H(u) \int_0^T \psi(v) d\beta^H(v) \right];$$

the space  $\mathcal{H}$  is thus what would be a RKHS at the level of the noise in  $L^2(0, T)$  which covariance is  $\int_0^{u \wedge v} \frac{\partial K}{\partial u}(u, s) \frac{\partial K}{\partial v}(v, s) ds$ .

In Hilbert spaces, we assume that  $W^H$  is the direct image by a Hilbert-Schmidt operator  $\Phi$  of a cylindrical fractional Wiener process on  $L^2$ , *i.e.*  $W^H = \Phi W_c^H$ . A cylindrical fractional Wiener process on a Hilbert space  $E$  is such that for every orthonormal basis  $(e_j)_{j \in \mathbb{N}}$  of  $L^2$  there exists independent fractional Brownian motions (fBm)  $(\beta_j^H(t))_{t \geq 0}$  such that  $W_c(t) = \sum_{j \in \mathbb{N}} \beta_j^H(t) e_j$ .

Stochastic integration with respect to fractional Wiener processes in a Hilbert space  $F$ , see for example [22], when integrands are deterministic is defined as above but for step functions multiplied by elements of the Hilbert space. It is such that a scalar product by an element of  $F$  is the one dimensional stochastic integral of the scalar product of the integrand. Operators  $K_T^*$  are still well defined when the RKHS  $\mathcal{H}$  is made of functions with values in  $F$ . Integrals of deterministic bounded operator valued integrands  $\Lambda$  from the Hilbert space  $E$  to  $F$  are defined for  $t$  positive as

$$\int_0^t \Lambda(s) dW^H(s) = \sum_{j \in \mathbb{N}} \int_0^t \Lambda(s) \Phi e_j d\beta_j^H(s) = \sum_{j \in \mathbb{N}} \int_0^t (K_t^* \Lambda(\cdot) \Phi e_j)(s) d\beta_j(s),$$

when  $(\Lambda(t))_{t \in [0, T]}$  is such that

$$\sum_{j \in \mathbb{N}} \int_0^T \|(K_T^* \Lambda(\cdot) \Phi e_j)(t)\|_F^2 dt < \infty.$$

Note that the duality relation (1.2) still holds, the integral is a Bochner integral, and that  $K_T^*$  commutes with the scalar product with an element of  $F$ . We now assume that  $E = L^2$  and that  $(e_j)_{j \in \mathbb{N}}$  is a complete orthonormal system. We may also check from (1.2) that the linear group  $(U(t))_{t \in \mathbb{R}}$  on a Sobolev space based on  $L^2$  commutes with  $K_T^*$ .

### 3. THE STOCHASTIC CONVOLUTION

In this section we present a few properties of the stochastic convolution.

When we consider particular saturated nonlinearities, the precise assumptions are given in the next section, we treat singular kernels and state our results in spaces of Hölder continuous functions. We thus make the following assumption

**Assumption (N1)**

$\Phi$  belongs to  $\mathcal{L}_2\left(L^2, H^{1+2(H+\alpha)}\right)$  with  $\left(\frac{1}{2} - H\right) \mathbb{1}_{H < \frac{1}{2}} < \alpha < (1 - H) \mathbb{1}_{H < \frac{1}{2}} + \mathbb{1}_{H \geq \frac{1}{2}}$ .



This assumption is used along with the fact that for  $\gamma$  in  $[0, 1)$  and  $t$  positive

$$(3.1) \quad \|U(t) - I\|_{\mathcal{L}_c(H^{1+2\gamma}, H^1)} \leq 2^{1-\gamma} |t|^\gamma;$$

it could be proved using the Fourier transform.

When we consider Kerr nonlinearities when the space dimension is such that  $d > 2$  we impose

**Assumption (N2)**

$$\Phi \in \mathcal{L}_2^{0,2} \text{ and } H > \frac{1}{2}.$$

In [6], the authors impose weaker assumptions on  $\Phi$ , namely  $\Phi \in \mathcal{L}_2^{0,1}$ , and check the required integrability of the stochastic convolution. It is more intricate for a fractional noise. This integrability follows from the Strichartz inequalities under (N2), however this assumption is certainly too strong.

Under (N1), the following result on the stochastic convolution holds.

**Lemma 3.1.** *The stochastic convolution  $Z : t \mapsto \int_0^t U(t-s) dW^H(s)$  is well defined. It has a modification in  $C_\infty^{H,0}$  and defines a  $C_\infty^{H,0}$ - random variable. Moreover, the direct images  $\mu^{Z,T,H'}$  of its law  $\mu^Z$  by the restriction on  $C_T^{H',0}$  for  $T$  positive and  $0 < H' < H$  are centered Gaussian measures.*

**Proof.** The stochastic convolution is well defined since for  $t$  positive

$$\begin{aligned} & \sum_{j \in \mathbb{N}} \int_0^t \| (K_t^* U(t-\cdot) \Phi e_j)(u) \|_{H^{1+2H}}^2 du \\ &= \sum_{j \in \mathbb{N}} \int_0^t \left\| U(-u) \Phi e_j K(t, u) + \int_u^t (U(-r) - U(-u)) \Phi e_j K(dr, u) \right\|_{H^{1+2H}}^2 du \\ &\leq 2 \|\Phi\|_{\mathcal{L}_2^{0,r+2H}}^2 \int_0^t K(t, u)^2 du + 2 \sum_{j \in \mathbb{N}} \int_0^t \left\| \int_u^t (U(-r) - U(-u)) \Phi e_j K(dr, u) \right\|_{H^{1+2H}}^2 du \\ &\leq 2(T_1 + T_2). \end{aligned}$$

Note that we used the continuous embedding of  $H^{1+2(H+\alpha)}$  into  $H^{1+2H}$ . The integral in  $T_1$  is equal to  $\mathbb{E}[(\beta^H(t))^2] = t^{2H}$ . Using (3.1), we obtain

$$T_2 \leq 4^{1-\alpha} \|\Phi\|_{\mathcal{L}_2^{0,r+2(H+\alpha)}}^2 c_H^2 \left( \frac{1}{2} - H \right)^2 \int_0^t \left( \int_u^t (r-u)^{H-\frac{3}{2}+\alpha} \left( \frac{r}{u} \right)^{H-\frac{1}{2}} dr \right)^2 du$$

thus

$$T_2 \leq 4^{1-\alpha} \|\Phi\|_{\mathcal{L}_2^{0,r+2(H+\alpha)}}^2 c_H^2 \left( \frac{1}{2} - H \right)^2 \int_0^t \left( \int_u^t (r-u)^{H-\frac{3}{2}+\alpha} dr \right)^2 du,$$

the integral is well defined since  $H - \frac{3}{2} + \alpha > -1$ . We finally obtain

$$T_2 \leq \frac{4^{\frac{1}{2}-\alpha} \|\Phi\|_{\mathcal{L}_2^{0,r+2(H+\alpha)}}^2}{H + \alpha} \left( \frac{c_H (H - \frac{1}{2})}{H - \frac{1}{2} + \alpha} \right)^2 t^{2H+2\alpha}.$$

Note that when  $H > \frac{1}{2}$ , the assumption on  $\alpha$  is not necessary, indeed the kernel is null on the diagonal and its derivative is integrable. We could obtain directly

$$T_2 \leq \|\Phi\|_{\mathcal{L}_2^{0,r+2(H+\alpha)}}^2 \int_0^t K(t, u)^2 du = \|\Phi\|_{\mathcal{L}_2^{0,r+2(H+\alpha)}}^2 t^{2H}.$$

We now prove that for any positive  $T$  and  $0 < H' < H$ ,  $Z$  has a modification in  $C_T^{H',0}$ . We prove that it has a modification which is in  $C^{H''}([0, T], H^1)$  for some  $H''$  such that

$H' < H'' < H$ . It will thus belong to  $C^{H',0}([0, T], H^1)$ . Note that, as we are dealing with a centered Gaussian process, upper bounds on higher moments could be deduced from an upper bound on the second order moment; see [5] for a proof in the infinite dimensional setting. It is therefore enough to show that there exists positive  $C$  and  $\gamma > H''$  such that for every  $(t, s) \in [0, T]^2$ .

$$\mathbb{E} [\|Z(t) - Z(s)\|_{H^1}^2] \leq C|t - s|^{2\gamma},$$

and then conclude with the Kolmogorov criterion.

When  $0 < s < t$ , we have

$$\begin{aligned} Z(t) - Z(s) &= U(s) (U(t - s) - I) \sum_{j \in \mathbb{N}} \int_0^T (K_T^* \mathbb{1}_{[0,t]}(\cdot) U(-\cdot) \Phi e_j)(w) d\beta_j(w) \\ &\quad + U(s) \sum_{j \in \mathbb{N}} \int_0^T ((K_t^* U(-\cdot) \Phi e_j)(w) - (K_s^* U(-\cdot) \Phi e_j)(w)) d\beta_j(w) \\ &= \tilde{T}_1(t, s) + \tilde{T}_2(t, s). \end{aligned}$$

We have

$$\begin{aligned} &\mathbb{E} \left[ \left\| \tilde{T}_1(t, s) \right\|_{H^1}^2 \right] \\ &\leq \|U(t - s) - I\|_{\mathcal{L}_c(H^{1+2(H+\alpha)}, H^1)}^2 \sum_{j \in \mathbb{N}} \int_0^T \left\| (K_T^* \mathbb{1}_{[0,t]}(\cdot) U(-\cdot) \Phi e_j)^2(w) \right\|_{H^{1+2(H+\alpha)}}^2 dw \\ &\leq C(T, H, \alpha) \|\Phi\|_{\mathcal{L}_2^{0, r+2(H+\alpha)}}^2 |t - s|^{2(H+\alpha)}, \end{aligned}$$

where  $C(T, H, \alpha)$  is a constant, and

$$\begin{aligned} &\mathbb{E} \left[ \left\| \tilde{T}_2(t, s) \right\|_{H^1}^2 \right] \\ &= \sum_{j \in \mathbb{N}} \int_0^T \|U(-u) \Phi e_j K(t, u) + \int_u^t (U(-r) - U(-u)) \Phi e_j K(dr, u) \\ &\quad - U(-u) \Phi e_j K(s, u) - \int_u^s (U(-r) - U(-u)) \Phi e_j K(dr, u)\|_{H^1}^2 du \\ &\leq \sum_{j \in \mathbb{N}} \left( \tilde{T}_{21}^j + \tilde{T}_{22}^j + \tilde{T}_{23}^j \right), \end{aligned}$$

where, using the fact that the kernel is triangular,

$$\begin{aligned} \tilde{T}_{21}^j &= \int_0^s \left\| U(-u) \Phi e_j (K(t, u) - K(s, u)) + \int_s^t (U(-r) - U(-u)) \Phi e_j K(dr, u) \right\|_{H^1}^2 du, \\ \tilde{T}_{22}^j &= 2 \int_s^t \|U(-u) \Phi e_j K(t, u)\|_{H^1}^2 du \\ \tilde{T}_{23}^j &= 2 \int_s^t \left\| \int_u^t (U(-r) - U(-u)) \Phi e_j K(dr, u) \right\|_{H^1}^2 du. \end{aligned}$$

We have

$$\begin{aligned} \tilde{T}_{21}^j &= \int_0^s \left\| \int_s^t U(-r) \Phi e_j K(dr, u) \right\|_{H^1}^2 du \\ &= \|\Phi e_j\|_{H^{s+\gamma}}^2 \int_0^s \left( \int_s^t |K(dr, u)| \right)^2 du \\ &= \|\Phi e_j\|_{H^{s+\gamma}}^2 \int_0^s (K(t, u) - K(s, u))^2 du \end{aligned}$$

thus

$$\begin{aligned} \tilde{T}_{21}^j &\leq \|\Phi e_j\|_{H^1}^2 \int_0^t (K(t, u) - K(s, u))^2 du \\ &\leq \|\Phi e_j\|_{H^1}^2 \mathbb{E} \left[ (\beta^H(t) - \beta^H(s))^2 \right] \\ &\leq \|\Phi e_j\|_{H^1}^2 |t - s|^{2H}, \end{aligned}$$

and

$$\begin{aligned} \tilde{T}_{22}^j &= 2 \|\Phi e_j\|_{H^{s+\gamma}}^2 \int_s^t K(t, u)^2 du \\ &= 2 \|\Phi e_j\|_{H^{s+\gamma}}^2 \int_s^t (K(t, u) - K(s, u))^2 du \end{aligned}$$

thus

$$\begin{aligned}\tilde{T}_{22}^j &\leq 2 \|\Phi e_j\|_{\mathbb{H}^1}^2 \int_0^t (K(t, u) - K(s, u))^2 du \\ &\leq 2 \|\Phi e_j\|_{\mathbb{H}^1}^2 |t - s|^{2H},\end{aligned}$$

finally the same computations as above shows that when  $H - \frac{3}{2} + \alpha > -1$  (used for integrability issue when  $H < \frac{1}{2}$ ), we have

$$\begin{aligned}\tilde{T}_{23}^j &\leq 4^{1-(H+\alpha)} \|\Phi\|_{\mathcal{L}_2^{0, r+2(H+\alpha)}}^2 c_H^2 \left(H - \frac{1}{2}\right)^2 \int_s^t \left( \int_u^t (r - u)^{2H - \frac{3}{2} + \alpha} \left(\frac{r}{u}\right)^{H - \frac{1}{2}} dr \right)^2 du \\ &\leq \frac{4^{\frac{1}{2} - (H+\alpha)} \|\Phi\|_{\mathcal{L}_2^{0, r+2(H+\alpha)}}^2}{2H + \alpha} \left( \frac{c_H (H - \frac{1}{2})}{2H - \frac{1}{2} + \alpha} \right)^2 (t - s)^{4H + 2\alpha}.\end{aligned}$$

Note that when  $H > \frac{1}{2}$  the kernel is null on the diagonal, its derivative has constant sign and it is integrable thus we can obtain without the assumption on  $\alpha$

$$\begin{aligned}\tilde{T}_{23}^j &\leq 4 \int_s^t \|\Phi e_j\|_{\mathbb{H}^1}^2 \left( \int_u^t |K(dr, u)| \right)^2 du \\ &\leq 4 \|\Phi e_j\|_{\mathbb{H}^1}^2 \int_s^t K(t, u)^2 du \\ &\leq 4 \|\Phi e_j\|_{\mathbb{H}^1}^2 \mathbb{E} [|\beta^H(t) - \beta^H(s)|^2] \\ &\leq 4 \|\Phi e_j\|_{\mathbb{H}^1}^2 |t - s|^{2H}.\end{aligned}$$

Thus  $Z$  admits a modification with  $H''$ -Hölder continuous sample paths with  $H' < H'' < H$ .

We now explain why  $Z$  has a modification which is in  $C([0, T], \mathbb{H}^{1+2H})$ . Since the group is an isometry we have

$$\begin{aligned}\|Z(t) - Z(s)\|_{\mathbb{H}^{1+2H}} &\leq \left\| (U(t - s) - I) \sum_{j \in \mathbb{N}} \int_0^T (K_T^* \mathbb{1}_{[0, t]}(\cdot) U(\cdot) \Phi e_j)(w) d\beta_j(w) \right\|_{\mathbb{H}^{1+2H}} \\ &\quad + \left\| \tilde{T}_2(t, s) \right\|_{\mathbb{H}^{1+2H}}.\end{aligned}$$

Since the group is strongly continuous and since, from the above,

$$\sum_{j \in \mathbb{N}} \int_0^T (K_T^* \mathbb{1}_{[0, t]}(\cdot) U(\cdot) \Phi e_j)(w) d\beta_j(w)$$

belongs to  $\mathbb{H}^{1+2H}$ , the first term of the right hand side goes to zero as  $s$  converges to  $t$ . Also, we may write

$$\left\| \tilde{T}_2(t, s) \right\|_{\mathbb{H}^{1+2H}} \leq \|Y(t) - Y(s)\|_{\mathbb{H}^{1+2H}}$$

where  $(Y(t))_{t \in [0, T]}$ , defined for  $t \in [0, T]$  by

$$Y(t) = \sum_{j \in \mathbb{N}} \int_0^T (K_t^* U(\cdot) \Phi e_j)(w) d\beta_j(w),$$

is a Gaussian process. We again conclude, with the same bounds for  $\tilde{T}_{21}^j$  and  $\tilde{T}_{22}^j$  and an upper of the order of  $(t - s)^{2H+2\alpha}$  for  $\tilde{T}_{23}^j$  and using the Kolmogorov criterion, that  $Y(t)$  admits a modification with continuous sample paths. Thus, for such a modification of  $Y$ ,  $Z$  has continuous sample paths.

The fact that  $\mu^{Z, T}$  are Gaussian measures follows from the fact that  $Z$  is defined as

$$\sum_{j \in \mathbb{N}} \int_0^t (K_T^* \mathbb{1}_{[0, t]}(\cdot) U(t - \cdot) \Phi e_j)(s) d\beta_j(s).$$

The law is Gaussian since the law of the action of an element of the dual is a pointwise limit of Gaussian random variables; see for example [13].

It is a standard fact to prove that the process defines a  $C_\infty^{H,0}$  random variable, see for example [13] for similar arguments. We use the fact that the process takes its values in a separable metrisable space.  $\square$

**Remark 3.2.** *The assumption on  $\alpha$  seems too strong to have the desired Hölder exponent. It is required only for integrability in the upper bounds of  $T_2$  and  $\tilde{T}_{23}^j$ . Also, the assumption that  $\Phi$  is Hilbert-Schmidt in a Sobolev space of exponent at least  $1 + 2H$  is only required in order that the convolution is a  $H^{1+2H}$  valued process. Indeed, there is a priori no reason that a Hölder continuous stochastic convolution gives rise to a Hölder continuous solution to the stochastic NLS equations. Hölder continuity of the deterministic free flow and convolution of the nonlinearity is obtained by assuming extra space regularity of the solution.*

In the following we always consider such a modification. The following lemma allows to characterize the RKHS of such Gaussian measures.

**Lemma 3.3.** *The covariance operator of  $Z$  on  $L^2(0, T; L^2)$  is given for  $h$  in  $L^2(0, T; L^2)$  by*

$$\mathcal{Q}h(t) = \sum_{j \in \mathbb{N}} \int_0^T \int_0^{t \wedge u} \left( K_T^* \mathbb{1}_{[0,t]}(\cdot) U(t - \cdot) \Phi e_j \right) (s) \left( (K_T^* \mathbb{1}_{[0,u]}(\cdot) U(u - \cdot) \Phi e_j) (s), h(u) \right)_{L^2} ds du,$$

when  $H > \frac{1}{2}$  we may write  $\mathcal{Q}h(t)$  as

$$c_H^2 \left( H - \frac{1}{2} \right)^2 \beta \left( 2 - 2H, H - \frac{1}{2} \right) \int_0^T \int_0^t \int_0^s |u-v|^{2H-2} U(t-v) \Phi \Phi^* U(u-s) h(s) du dv ds.$$

Also, for  $T$  positive and  $0 < H' < H$ , the RKHS of  $\mu^{Z,T,H'}$  is  $\text{im } \mathcal{Q}^{\frac{1}{2}}$  with the norm of the image structure. It is also  $\text{im } \mathcal{L}$  where  $\mathcal{L}$  is defined for  $h$  in  $L^2(0, T; L^2)$  by

$$\mathcal{L}h(t) = \sum_{j \in \mathbb{N}} \int_0^t (K_T^* \mathbb{1}_{[0,t]}(\cdot) U(t - \cdot) \Phi e_j) (s) (h(s), e_j)_{L^2} ds.$$

**Proof.** We may first check with the same computations as those used in Lemma 3.1 that  $\mathcal{L}$  is well defined and that for  $h$  in  $L^2(0, T; L^2)$ ,  $\mathcal{L}h$  belongs to  $L^2(0, T; L^2)$ . Take  $h$  and  $k$  in  $L^2(0, T; L^2)$ , we have

$$\begin{aligned} & \mathbb{E} \left[ \int_0^T (Z(u), h(u))_{L^2} du \int_0^T (Z(t), k(t))_{L^2} dt \right] \\ &= \sum_{j \in \mathbb{N}} \mathbb{E} \left[ \int_0^T \int_0^T \left( \int_0^T (K_T^* \mathbb{1}_{[0,u]}(\cdot) U(u - \cdot) \Phi e_j) (s) d\beta_j(s), h(u) \right)_{L^2} \right. \\ & \quad \left. \left( \int_0^T (K_T^* \mathbb{1}_{[0,t]}(\cdot) U(t - \cdot) \Phi e_j) (v) d\beta_j(v), k(t) \right)_{L^2} \right] \\ &= \int_0^T (\mathcal{Q}h(t), k(t))_{L^2} dt \end{aligned}$$

where  $\mathcal{Q}$  is defined in the lemma. The result for  $H > \frac{1}{2}$  is obtained with the particular form of the inner product in  $\mathcal{H}$  for such values of  $H$ .

Checking that for  $k$  in  $L^2(0, T; L^2)$ ,

$$\mathcal{L}^* k(s) = \sum_{j \in \mathbb{N}} \int_s^T ((K_T^* \mathbb{1}_{[0,t]}(\cdot) U(t - \cdot) \Phi e_j) (s), k(t))_{L^2} e_j dt,$$

we obtain that  $\mathcal{Q} = \mathcal{L}\mathcal{L}^*$ .

We may thus deduce, see for example [13], that the RKHS of  $\mu^{Z,T,H'}$  is also  $\text{im } \mathcal{L}$  with the norm of the image structure. It is indeed the RKHS of the direct image of  $\mu^{Z,T,H'}$  on

$L^2(0, T; L^2)$  but it is standard fact, see for example [5, 13], that the two measures have same RKHS.  $\square$

When we impose (N2) we can prove as above that the stochastic convolution  $Z$  has a modification in  $C([0, \infty); H^2)$  embedded with the projective limit topology letting the time interval go to infinity. Thus from the Sobolev embeddings, for any  $T$  positive and  $(r(p), p)$  an admissible pair,  $Z$  belongs to  $X^{(T, p)} = C([0, T]; H^1) \cap L^{r(p)}(0, T; W^{1, p})$ . As mentioned previously, this space is considered to do the fixed point that allows to prove the local well-posedness for Kerr nonlinearities. We may also check

**Lemma 3.4.**  *$Z$  defines a  $C([0, \infty); H^2)$ -random variable. The law of its projections  $\mu^{Z, T}$  on  $C([0, T]; H^2)$  for  $T$  positive is a centered Gaussian measure whose RKHS is  $\text{im } \mathcal{L}$ .*

We now deduce the following results that we will push forward to obtain results for the solution of the SPDE.

**Proposition 3.5.** *The direct image measures for  $\epsilon$  positive of  $x \mapsto \sqrt{\epsilon}x$  on  $C_\infty^{H, 0}$ , respectively  $C([0, \infty); H^2)$ , satisfy a LDP of speed  $\epsilon$  and good rate function*

$$I^Z(f) = \frac{1}{2} \inf_{h \in L^2(0, \infty; L^2): \mathcal{L}(h)=f} \left\{ \|h\|_{L^2(0, \infty; L^2)}^2 \right\}.$$

**Proof.** From a general result on LDP for Gaussian measures on Banach spaces, see [12], and the above lemma, we know that for  $T$  positive and  $0 < H' < H$ , the direct images of  $\mu^{Z, T, H'}$  by the mapping  $x \mapsto \sqrt{\epsilon}x$  satisfy a LDP of speed  $\epsilon$  and good rate function

$$I^{Z, T, H'}(f) = \frac{1}{2} \inf \left\{ \|h\|_{\text{im } \mathcal{L}}^2 : f = \mathcal{L}h \right\}$$

with the convention that  $\inf \emptyset = \infty$ . We conclude letting  $T$  go to infinity and  $H'$  to  $H$  using Dawson-Gartner's theorem for projective limits, see for example [11], and Lebesgue's dominated convergence theorem. The same is true under (N2) when we work in  $C([0, \infty); H^2)$ .  $\square$

**Proposition 3.6.** *Under (N1) the support of the measure  $\mu^Z$  is given by*

$$\text{supp } \mu^Z = \overline{\text{im } \mathcal{L}^{C_\infty^{H, 0}}},$$

*under (N2) the same result holds replacing  $C_\infty^{H, 0}$  by  $C([0, \infty); H^2)$ .*

**Proof.** Let us give the argument when we have the assumption (N1), the argument under (N2) is the same.

From the characterization of the RKHS of the centered Gaussian measure  $\mu^{Z, T, H'}$  for  $T$  positive and  $0 < H' < H$  and Theorem (IX, 2; 1) in [3], we obtain that the support of the measure  $\mu^{Z, T, H'}$  is such that

$$\text{supp } \mu^{Z, T, H'} = \overline{\text{im } \mathcal{L}_T^{C_T^{H', 0}}}.$$

From the definition of the image measure we have that

$$\mu^Z \left( p_{T, H'}^{-1} \left( \overline{\text{im } \mathcal{L}_T^{C_T^{H', 0}}} \right) \right) = \mu^{Z, T, H'} \left( \overline{\text{im } \mathcal{L}_T^{C_T^{H', 0}}} \right) = 1,$$

where  $p_{T, H'}$  denotes the projection of  $C_\infty^{H, 0}$  into  $C_T^{H', 0}$ . It follows that

$$\text{supp } \mu^Z \subset \bigcap_T p_{T, H'}^{-1} \left( \overline{\text{im } \mathcal{L}_T^{C_T^{H', 0}}} \right) = \overline{\text{im } \mathcal{L}^{C_\infty^{H, 0}}}.$$

It then suffices to show that  $\text{im } \mathcal{L} \subset \text{supp } \mu^Z$ . Suppose that  $x \notin \text{supp } \mu^Z$ , then there exists a neighborhood  $V$  of  $x$  in  $C_\infty^{H,0}$  which is a neighborhood of  $x$  in  $C_T^{H',0}$  for  $T$  large and  $H'$  sufficiently close to  $H$  such that  $\mu^Z(V) = 0$ . Since the support of  $\mu^{Z,T,H'}$  is the closure of  $\text{im } \mathcal{L}$  for the topology of  $C_T^{H',0}$ ,  $V \cap \text{im } \mathcal{L} = \emptyset$  and  $x \notin \text{im } \mathcal{L}$ .  $\square$

#### 4. LOCAL WELL-POSEDNESS OF THE CAUCHY PROBLEM

We consider the Cauchy problem

$$(4.1) \quad \begin{cases} i du = (\Delta u + f(u)) dt + dW^H \\ u(0) = u_0. \end{cases}$$

We consider two cases. In the first case we assume (N1),  $u_0 \in H^{1+2H}$  and

**Assumption (NL)**

- (i)  $f$  is Lipschitz on the bounded sets of  $H^{1+2H}$
- (ii)  $f(0) = 0$ .

In the second case we assume (N2),  $u_0 \in H^1$  and  $f$  is a Kerr nonlinearity.

We first recall the following important fact. Let us denote by  $v^{u_0}(z)$  the solution of

$$(4.2) \quad \begin{cases} i \frac{dv}{dt} = \Delta v + f(v - iz) \\ u(0) = u_0. \end{cases}$$

where  $z$  is a function of  $C_\infty^{H,0}$  (repectively  $C([0, \infty), H^2)$ ) and define  $\mathcal{G}^{u_0}$  the mapping

$$\mathcal{G}^{u_0} : z \mapsto v^{u_0}(z) - iz.$$

Then we may check that the solution  $u^{\epsilon, u_0}$  of (4.1) is such that  $u^{\epsilon, u_0} = \mathcal{G}^{u_0}(\sqrt{\epsilon}Z)$  where  $Z$  is the stochastic convolution.

We may now check with a fixed point argument the following result.

**Theorem 4.1.** *Assume that the initial datum  $u_0$  is  $\mathcal{F}_0$  measurable and belongs to  $H^{1+2H}$  (repectively  $H^1$ ); then there exists a unique solution to (1.2) with continuous  $H^{1+2H}$  (respectively  $H^1$ ) valued paths. The solution is defined on a random interval  $[0, \tau^*(u_0, \omega))$  where  $\tau^*(u_0, \omega)$  is either  $\infty$  or a finite blow-up time.*

In the next two section we state sample paths LDPs and support theorems. We start with the first set of assumptions and state a result in a space of Hölder continuous sample paths with any value of the Hurst parameter. In the last section we consider the case of Kerr nonlinearities and restrict ourselves to the case where  $H > \frac{1}{2}$ .

#### 5. THE CASE OF A NONLINEARITY SATISFYING (NL)

According to (NL) solutions may blow up in finite time. We shall proceed as in [13] to define proper path spaces where we can state the LDP and support result; see the reference for more details. However, we consider here a space where paths are  $H'$ -Hölder continuous with values in  $H^1$  on compact time intervals before the blow-up time where  $0 < H' < H$ . We add a point  $\Delta$  to the space  $H^{1+2H}$  and embed the space with the topology such that its open sets are the open sets of  $H^{1+2H}$  and the complement in  $H^{1+2H} \cup \{\Delta\}$  of the closed bounded sets of  $H^{1+2H}$ . The set  $C([0, \infty); H^{1+2H} \cup \{\Delta\})$  is then well defined. We denote the blow-up time of  $f$  in  $C([0, \infty); H^{1+2H} \cup \{\Delta\})$  by

$\mathcal{T}(f) = \inf\{t \in [0, \infty) : f(t) = \Delta\}$ , with the convention that  $\inf \emptyset = \infty$ .

We also define the following spaces

$$C_T^{H'} = C([0, T]; H^{1+2H}) \cap C^{H'}([0, T]; H^1)$$

and

$$C_T^{H',0} = C([0, T]; H^{1+2H}) \cap C^{H',0}([0, T]; H^1).$$

When equipped with the norm which is the supremum of the norms of the two Banach spaces intersected they are Banach spaces. The latter space is separable.

For measurability issue, we define

$$C_\infty^{H,0} = \bigcap_{T>0, 0<H'<H} C_T^{H',0}$$

equipped with the projective limit topology. It is a separable metrisable space. We also define

$$\begin{aligned} \mathcal{E}^H(H^1) = & \left\{ f \in C([0, \infty); H^{1+2H} \cup \{\Delta\}) : f(t_0) = \Delta \Rightarrow \forall t \geq t_0, f(t) = \Delta; \right. \\ & \left. \forall T < \mathcal{T}(f), \forall 0 < H' < H, f \in C^{H'}([0, T]; H^1) \right\}. \end{aligned}$$

Here  $\Delta$  acts as a cemetery. It is endowed with the topology defined by the neighborhood basis

$$V_{T,R,H'}(\varphi_1) = \left\{ \varphi \in \mathcal{E}^H(H^1) : \mathcal{T}(\varphi) > T, \|\varphi_1 - \varphi\|_{C_T^{H'}} \leq R \right\},$$

of  $\varphi_1$  in  $\mathcal{E}^H(H^1)$  given  $T < \mathcal{T}(\varphi_1)$  and  $R$  positive. The space is also a Hausdorff topological space and thus we may consider applying the Varadhan contraction principle.

In order to push forward the results of section 3 we use the following result.

**Lemma 5.1.** *The mapping*

$$\begin{aligned} C_\infty^{H,0} & \rightarrow \mathcal{E}^H(H^1) \\ z & \mapsto \mathcal{G}^{u_0}(z) \end{aligned}$$

*is continuous.*

**Proof.** This could be done by revisiting the fixed point argument, this time in  $C_{T^*}^{H'}$  for  $T^*$  small enough depending on the norm of the initial data and  $z$  in  $C_T^{H'}$  for some fixed  $T$  and some  $H' < H$  fixed. Though with different norms, the remaining of the argument allowing to prove the continuity of  $v^{u_0}(z)$  with respect to  $z$ , detailed in [6], holds. In the computations we use (3.1) in order to treat the Hölder norms.  $\square$

Note that Lemma 3.1 and 5.1 give that  $u^{1,u_0}$  defines a  $\mathcal{E}^H(H^1)$  random variable.

Let us now study large deviations for the laws  $\mu^{u^\epsilon, u_0}$  on  $\mathcal{E}^H(H^1)$  of the mild solutions  $u^{\epsilon, u_0}$  of

$$(5.1) \quad \begin{cases} \epsilon du - (\Delta u + f(u))dt = \sqrt{\epsilon} dW^H, \\ u(0) = u_0 \in H^{1+2H}. \end{cases}$$

We may now deduce from Lemma 3.3 and 5.1, the fact that  $(\mathcal{G}^{u_0} \circ \mathcal{L})(\cdot) = \mathbf{S}(u_0, \cdot)$ , and the Varadhan contraction principle the following theorem.

**Theorem 5.2.** *The laws  $\mu^{u^\epsilon, u_0}$  on  $\mathcal{E}^H(H^1)$  satisfy a LDP of speed  $\epsilon$  and good rate function*

$$I^{u_0}(w) = \frac{1}{2} \inf_{h \in L^2(0, \infty; L^2) : \mathbf{S}(u_0, h) = w} \left\{ \|h\|_{L^2(0, \infty; L^2)}^2 \right\},$$

where  $\mathbf{S}(u_0, h)$  denotes the mild solution in  $\mathcal{E}^H(\mathbf{H}^1)$  of the following control problem

$$(5.2) \quad \begin{cases} i \frac{\partial u}{\partial t} - (\Delta u + f(u)) = \Phi \dot{K}h, \\ u(0) = u_0 \in \mathbf{H}^{1+2H}, \quad h \in L^2(0, \infty; L^2); \end{cases}$$

it is called the skeleton. Only the integral, or the integral in the mild formulation, of the right hand side is defined; it is by means of the duality relation.

**Remark 5.3.** We could also prove a uniform LDP as for example in [14].

The characterization of the support follows with the same arguments as in [13]. We recall the proof for the sake of completeness.

**Theorem 5.4.** The support of the law  $\mu^{u^1, u_0}$  on  $\mathcal{E}^H(\mathbf{H}^1)$  is given by

$$\text{supp } \mu^{u^1, u_0} = \overline{\text{im } \mathbf{S}^{\mathcal{E}^H(\mathbf{H}^1)}}.$$

**Proof.** We use the continuity of  $\mathcal{G}$ . Indeed, since  $\mathcal{G}^{u_0}(\text{im } \mathcal{L}) \subset \overline{\mathcal{G}^{u_0}(\text{im } \mathcal{L})}^{\mathcal{E}^H(\mathbf{H}^1)}$ ,  $\text{im } \mathcal{L} \subset (\mathcal{G}^{u_0})^{-1} \left( \overline{\mathcal{G}^{u_0}(\text{im } \mathcal{L})}^{\mathcal{E}^H(\mathbf{H}^1)} \right)$ . Because  $\mathcal{G}^{u_0}$  is continuous, the right hand side is a closed set of  $C_\infty^{H,0}$  and from Proposition 3.6,

$$\text{supp } \mu^Z \subset (\mathcal{G}^{u_0})^{-1} \left( \overline{\text{im } (\mathcal{G}^{u_0} \circ \mathcal{L})}^{\mathcal{E}^H(\mathbf{H}^1)} \right),$$

and

$$\mu^Z \left( (\mathcal{G}^{u_0})^{-1} \left( \overline{\text{im } \mathbf{S}(u_0)}^{\mathcal{E}^H(\mathbf{H}^1)} \right) \right) = 1,$$

thus

$$\text{supp } \mu^u \subset \overline{\text{im } \mathbf{S}(u_0)}^{\mathcal{E}^H(\mathbf{H}^1)}.$$

Suppose that  $x \notin \text{supp } \mu^{u^1, u_0}$ , there exists a neighborhood  $V$  of  $x$  in  $\mathcal{E}^H(\mathbf{H}^1)$  such that  $\mu^{u^1, u_0}(V) = \mu^Z \left( (\mathcal{G}^{u_0})^{-1}(V) \right) = 0$ , consequently  $(\mathcal{G}^{u_0})^{-1}(V) \cap \text{im } \mathcal{L}$  is empty and  $x \notin \text{im } \mathbf{S}(u_0)$ . This gives the reverse inclusion.  $\square$

## 6. THE CASE OF KERR NONLINEARITIES

In this section we consider Kerr nonlinearities when  $d \geq 2$  and  $\sigma < \frac{2}{d-2}$ .

This time, we will not state a result in a space of Hölder continuous functions with values in  $\mathbf{H}^1$ . We would need that the convolution which involves the nonlinearity is Hölder continuous. Thus, in order to use (3.1), we would have to compute the Sobolev norm of the nonlinearity in some space  $\mathbf{H}^{1+2\gamma}$  where  $\gamma$  is positive.

**Remark 6.1.** In the case where  $H < \frac{1}{2}$ , we could however state a weaker result than in the previous section imposing that  $u_0 \in \mathbf{H}^1$  and  $\Phi \in \mathcal{L}_2^{0,2+\alpha}$ . The corresponding fixed point could be conducted in  $C^{H'}([0, T]; \mathbf{H}^{1-2H}) \cap C([0, T]; \mathbf{H}^1) \cap L^{r(p)}(0, T; W^{1,p})$  where  $(r(p), p)$  is an admissible pair and  $0 < H' < H$  and uses the Strichartz inequalities. Indeed, from the Sobolev embeddings, the stochastic convolution has a modification in  $C([0, T]; \mathbf{H}^2) \cap C^{H'}([0, T]; \mathbf{H}^{2-2H})$  and thus belong to the desired space.

Let us return to the case where  $H > \frac{1}{2}$ . Since under (N2) we know that the stochastic convolution  $Z$  has a modification in  $X^{(T,p)}$ , we can directly use the continuity of the solution with respect to the stochastic convolution of [13] and repeat the remaining of the



arguments. Thus, for initial data in  $H^1$ , we may state a LDP and support result in the space  $\mathcal{E}_\infty$  defined as

$$\mathcal{E}_\infty = \left\{ f \in C([0, \infty); H^1 \cup \{\Delta\}) : f(t_0) = \Delta \Rightarrow \forall t \geq t_0, f(t) = \Delta; \right. \\ \left. \forall T < \mathcal{T}(f), \forall p \in \left[2, \frac{2d}{d-2}\right), f \in L^{r(p)}(0, T; W^{1,p}) \right\}.$$

When  $d = 2$  or  $d = 1$  we write  $p \in [2, \infty)$ . The space is embedded with the topology defined by the neighborhood basis

$$W_{T,p,R}(\varphi_1) = \{\varphi \in \mathcal{E}_\infty : \mathcal{T}(\varphi) \geq T, \|\varphi_1 - \varphi\|_{X^{(T,p)}} \leq R\},$$

for  $\varphi_1$  in  $\mathcal{E}_\infty$ .

**Theorem 6.2.** *The laws  $\mu^{u^\epsilon, u_0}$  on  $\mathcal{E}_\infty$  satisfy a LDP of speed  $\epsilon$  and good rate function*

$$I^{u_0}(w) = \frac{1}{2} \inf_{h \in L^2(0, \infty; L^2) : \mathbf{S}(u_0, h) = w} \left\{ \|h\|_{L^2(0, \infty; L^2)}^2 \right\},$$

where  $\mathbf{S}(u_0, h)$  is the mild solution of

$$(6.1) \quad \begin{cases} i \frac{\partial u}{\partial t} - (\Delta u + \lambda |u|^{2\sigma} u) = \Phi \dot{K} h, \\ u(0) = u_0 \in H^1, \quad h \in L^2(0, \infty; L^2); \end{cases}$$

**Theorem 6.3.** *The support of the law  $\mu^{u^1, u_0}$  on  $\mathcal{E}_\infty$  is given by*

$$\text{supp } \mu^{u^1, u_0} = \overline{\text{im } \mathbf{S}^\infty}.$$

## REFERENCES

- [1] Alos, E.; Mazet, O.; Nualart, D. Stochastic calculus with respect to Gaussian processes. *Ann. Probab.* **1999**, *29*, 766-801.
- [2] Alos, E.; Nualart, D. Stochastic integration with respect to the fractional Brownian motion. *Stochastics Stochastics Rep.* **2003**, *75*, 129-152.
- [3] Badrikian, A.; Chevet, S. *Mesures Cylindriques, Espaces de Wiener et Fonctions Aléatoires Gaussiennes*. Lecture Notes in Math. **1974**, *379*.
- [4] Coutin, L.; Qian, Z. Stochastic analysis, rough path analysis and fractional Brownian motions. *Probab. Theory Relat. Fields* **2002**, *122*, 108-140.
- [5] Da Prato, G.; Zabczyk, J. *Stochastic Equations in Infinite Dimensions*; Encyclopedia of Mathematics and its Applications, Cambridge University Press: Cambridge, England, 1992.
- [6] de Bouard, A.; Debussche, A. The Stochastic Nonlinear Schrödinger Equation in  $H^1$ . *Stochastic Anal. Appl.* **2003**, *21*, 97-126.
- [7] de Bouard, A.; Debussche, A. On the effect of a noise on the solutions of the focusing supercritical nonlinear Schrödinger equation. *Probab. Theory Relat. Fields* **2002**, *123*, 76-96.
- [8] Decreusfond, L. Stochastic integration with respect to Volterra processes. *Ann. Inst. H. Poincaré Probab. Statist.* **2005**, *45*, 123-149.
- [9] Decreusfond, L.; Üstünel, A. Stochastic analysis of the fractional Brownian motion. *Potential Anal.* **1997**, *10*, 177-214.
- [10] Debussche, A.; Gautier, E. Small noise asymptotic of the timing jitter in soliton transmission. Preprint IRMAR and CREST **2005**.
- [11] Dembo, A.; Zeitouni, O. *Large deviation techniques and applications* (2-nd edition); Appl. of Math. 38, Springer-Verlag: New York, 1998.
- [12] Deuschel, J.D.; Stroock, D.W. *Large Deviations*; Pure and Appl. Math., Academic Press: New-York, 1986.
- [13] Gautier, E. Large deviations and support results for nonlinear Schrödinger equations with additive noise and applications. *ESAIM: Probability and Statistics* **2005**, *9*, 74-97.
- [14] Gautier, E. Uniform large deviations for the nonlinear Schrödinger equation with multiplicative noise. *Stochastic Process. Appl.* **2005**, *115*, 1904-1927.
- [15] Gautier, E. Exit from a neighborhood of zero for weakly damped stochastic nonlinear Schrödinger equations. Preprint IRMAR and CREST **2005**.

- [16] Ledoux, M.; Lyons, T.; Qian, Z. Lévy area of Wiener processes on Banach spaces. *Ann. Probab.* **2002**, *30*, 546-578.
- [17] Ledoux, M.; Qian, Z.; Zhang, T. Large deviations and support theorem for diffusion process via rough paths. *Stochastic Process. Appl.* **2002**, *102*, 265-283.
- [18] Lejay, A.; Gubinelli, M.; Tindel, S. Young integrals and SPDEs. Preprint IECN, Nancy **2004** <http://arxiv.org/abs/math.PR/0407294>.
- [19] Lyons, T.J. Differential equations driven by rough signals. *Rev. Mat. Iberoamericana* **1998**, *14*, 215-310.
- [20] Millet, A.; Sanz-Solé, M. Large deviations for rough paths of the fractional Brownian motion. *Ann. Inst. H. Poincaré Probab. Statist.* **2006**, *42*, 245-271.
- [21] Sulem, C.; Sulem, P.L. *The Nonlinear Schrödinger Equation, Self-Focusing and Wave Collapse*; Appl. Math. Sci. 139, Springer-Verlag: New York, 1999.
- [22] Tindel, S.; Tudor, C.A.; Viens, F. Stochastic evolution equations with fractional Brownian motion. *Probab. Theory Related Fields* **2003**, *127*, 186-204.